



## Universal aspects of curved, flat, and stationary-state Kardar-Parisi-Zhang statistics

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Motivated by the recent exact solution of the *stationary-state* Kardar-Parisi-Zhang (KPZ) statistics by Imamura and Sasamoto [Phys. Rev. Lett. **108**, 190603 (2012)], as well as a precursor experimental signature unearthed by Takeuchi [Phys. Rev. Lett. **110**, 210604 (2013)], we establish here the universality of these phenomena, examining scaling behaviors of directed polymers in a random medium, the stochastic heat equation with multiplicative noise, and kinetically roughened KPZ growth models. We emphasize the value of cross KPZ-class universalities, revealing crossover effects of experimental relevance. Finally, we illustrate the great utility of KPZ scaling theory by an optimized numerical analysis of the Ulam problem of random permutations.

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Extremal paths through random energy landscapes, i.e., “directed polymers in random media” (DPRM), have long been a topic of great interest to statistical physicists, condensed matter theorists, and mathematicians alike [1]. In two dimensions, the exact limit distributions of the DPRM problem [2] are of the celebrated Tracy-Widom (TW) type [3], best known perhaps from random matrix theory [4,5], as well as the famous Ulam problem of random permutations [6–11]. Nevertheless, in the bulk, the statistics of these extremal trajectories remains a challenging, rich, and quite difficult problem [12]. In all dimensions, however, the constrained free-energy  $F(\mathbf{x}, t)$  of these directed, extremal paths is dictated by the stochastic partial differential equation [13] of Kardar, Parisi, and Zhang (KPZ):

$$\partial_t F = v \partial_x^2 F + \frac{1}{2} \lambda (\partial_x F)^2 + \sqrt{D} \eta,$$

where  $v$ ,  $\lambda$ , and  $D$  are system-dependent parameters, the last setting the strength of the *additive* stochastic noise  $\eta$ . Rigorous mathematical approaches, by contrast, have focused on the related stochastic heat equation (SHE) with *multiplicative* noise, obtained from KPZ via a Hopf-Cole transformation,  $F = \frac{2v}{\lambda} \ln Z$ ; here, with  $\sqrt{\epsilon} = \lambda \sqrt{2D}/v^3$  and Itô interpretation, this becomes an elemental, rescaled version [14] of the SHE:

$$\partial_t Z = \nabla^2 Z + \frac{\epsilon}{2} Z + \sqrt{\epsilon} Z \eta,$$

governing the DPRM partition function  $Z(\mathbf{x}, t)$ , a much better behaved quantity, well regularized in the UV.

Inspired by Imamura and Sasamoto’s exact solution [15] of the KPZ equation *stationary-state* (SS) statistics, as well as Takeuchi’s subsequent reexamination [16] of the four-force KPZ turbulent liquid-crystal experiments [17], we focus here on the (1 + 1)-dimensional DPRM/SHE, making manifest its connection to the underlying Baik-Rains (BR)  $F_0$  limit distribution [18] relevant to this KPZ class. We examine, too, DPRM scaling phenomena in the pt-line and pt-pt configurations, analog of KPZ stochastic growth in *flat* and *curved* geometries, governed by Tracy-Widom distributions appropriate to Gaussian orthogonal and unitary matrix ensembles; i.e., TW-GOE and GUE, respectively. Our DPRM/SHE results in this regard provide the final installment of the KPZ triumvirate, complementing prior numerical confirmation of Tracy-Widom universality which had focused initially on kinetic roughening models such as polynuclear growth (PNG) [19], as well as single-step [20],

or totally asymmetric exclusion processes [21]. In fact, we go further here in the DPRM/SHE context, making additional suggestions regarding experimental signatures of KPZ-class statistics, examining the interplay of transient and SS regimes and, finally, in the pt-pt TW-GUE setting, illustrate the potency of KPZ scaling theory [22] to inform key aspects of the purely mathematical Ulam problem. We thus return the favor here, reiterating the very fruitful dialog between KPZ physics and TW-GUE mathematics.

In Fig. 1, for starters, we compare to TW-GUE the shifted, rescaled distributions of our *curved* KPZ-class models [12], among them the (i) pt-pt  $g_1$  DPRM; i.e., extremal paths in the  $xy$  plane, traveling in the [01] direction, Gaussian random energies  $\epsilon_i$  with zero mean and variance  $\frac{1}{4}$  on the sites of the square lattice; side-steps are permitted, but incur a microscopic, elastic energy cost  $\gamma = 1$ . We consider extremal paths that commence at the origin, constrained to terminate  $t = 300$  steps later on the axis. The  $g_1$  DPRM trace in Fig. 1 represents an average over  $10^8$  realizations of the random energy landscape. Similarly, (ii) pt-pt w/e DPRM—here, the random site energies are exponentially distributed,  $p(\epsilon) = e^{-\epsilon}$ , and the paths, on average, cut diagonally through the plane in the [11] direction; at each step, there are two possibilities, either vertically (i.e., in  $y$  direction) or horizontally (in  $x$  direction). There is no elastic energy cost in this model; the total energy of the trajectory simply the sum,  $F(t) = \sum \epsilon_i$ , of the random energies collected traveling from  $(0, 0)$  to  $(t, t)$ . For our w/e DPRM, we averaged over  $4 \times 10^8$  paths of length  $t = 400$ . Next, the numerically demanding (iii) constrained SHE-Itô integrations, with KPZ parameter  $\sqrt{\epsilon} = 10$ ; we evolved the system through the forward and rear light cones of the fixed initial and final points of the trajectory, averaging over  $10^7$  runs, considering paths of 4000 steps, with time increment  $\delta t = 0.05$ .

In the case of the pt-pt DPRM models, we have extracted, from a first principles Krug-Meakin (KM) finite-size scaling analysis [23], the nonuniversal, *system-dependent* KPZ parameters  $f_\infty$ ,  $A = D/2v$ , and  $\lambda$ . Note that the KM toolbox yields the asymptotic free-energy per step  $f_\infty$  and the *product*  $A\lambda$ , so that  $\lambda$ , itself, must be fixed by the parabolic DPRM free-energy profile [24]. Interestingly, in the following, only  $f_\infty$  and the combination  $\Gamma = \frac{1}{2} A^2 \lambda$  are needed, which we record in Table I. An alternative approach, which highlights the utility of cross KPZ-class studies emphasized in this Rapid Communication, involves fixing the KPZ scaling parameter

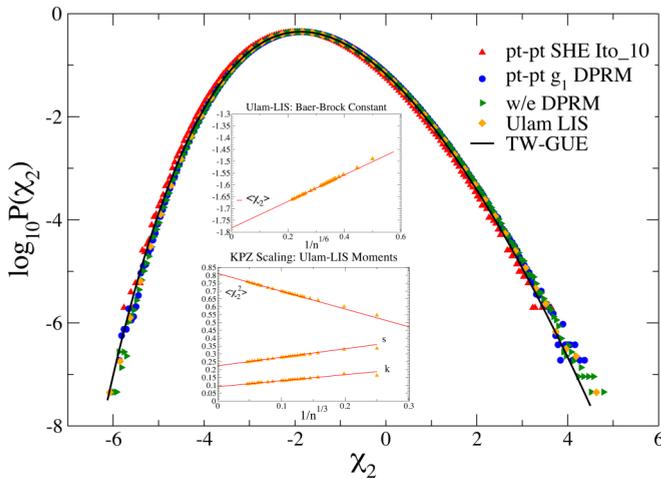


FIG. 1. (Color online) Universal limit distributions: DPRM/SHE, Ulam-LIS, and TW-GUE. Insets: Extraction, via KPZ scaling theory, of Ulam-LIS asymptotic moments and Baer-Brock constant.

$\Gamma$  directly from a fit, e.g., to the TW-GOE variance and then *using it in the service of* both the TW-GUE limit distribution, our immediate focus, as well as stationary-state Baik-Rains probability distribution function (PDF), which we consider later. This we do, for illustrative purposes, in the case of SHE-Itô integration. With  $\beta_{\text{KPZ}} = \frac{1}{3}$  the known [25] DPRM free-energy fluctuation exponent, the central KPZ scaling ansatz reads  $F_{\text{pt-pt}} = f_{\infty}t + (\Gamma t)^{\beta} \chi_2$  with  $\chi_2$  the underlying, order one Tracy-Widom GUE statistical variable. Thus, the pt-pt DPRM/SHE distributions of Fig. 1 have been cast in terms of  $\chi_2$ , the canonical quantity intrinsic to the *curved* KPZ class in this dimension. While crafting Fig. 1, we have relied upon the model-dependent *asymptotic* values for the distribution mean and variance,  $\langle \chi_2^2 \rangle = \lim_{t \rightarrow \infty} \langle \delta F^2 \rangle / (\Gamma t)^{2\beta}$ ; our curve fits for the latter assuming the characteristic  $t^{-2\beta}$  DPRM finite-time correction. Our most accurate values for these limit distribution characteristics are obtained via the Gaussian and exponential polymers—see Table I; compare, too, our model estimates for the distribution skewness  $s = \langle \delta F^3 \rangle_c / \langle \delta F^2 \rangle_c^{3/2}$  and kurtosis  $k = \langle \delta F^4 \rangle_c / \langle \delta F^2 \rangle_c^2$ . The accepted values [26] for these TW-GUE quantities are  $(\langle \chi_2 \rangle, \langle \chi_2^2 \rangle, s_2, k_2) = (-1.771\,087, 0.813\,195, 0.224\,084, 0.093\,448)$ . Our pt-pt DPRM-SHE results in Fig. 1, stacked up against the exact, known 1+1 curved KPZ-class TW-GUE distribution, provide an interesting extremal path counterpoint to efforts on radial two-dimensional Eden clusters [27,28] and kinetic roughening in droplet geometries [19,21,29], the latter studies involving PNG, single-step, ballistic deposition (BD), [30] as well as corner growth in a “restricted-solid-on-solid” (RSOS) model.

TABLE I. pt-pt DPRM/SHE: Model-dependent parameters; estimates of *universal* 1+1 curved KPZ-class TW-GUE quantities.

KPZ system	$f_{\infty}$	$\Gamma^{\beta}$	$\langle \chi_2 \rangle$	$\langle \chi_2^2 \rangle$	$ s_2 $	$k_2$
$g_5$ , DPRM	-0.498021	0.50633	-1.77097	0.8167	0.2304	0.0924
w/e DPRM	-2.0003	1.9882	-1.77102	0.8116	0.2227	0.0908
SHE-Itô	2.48533	0.13345	-1.7908	0.8121	0.2263	0.0893

Finally, in homage to the pioneering efforts of Baer and Brock [7], as well as the extraordinary numerical work of Odlyzko and Rains [8] a generation later, we have performed, in a post-KPZ/TW context, our own analysis of the Ulam problem, studying the fluctuating statistics of the length  $\ell_n$  of the longest increasing subsequence (LIS) in a permutation of  $n$  integers. It was established early on, with much heavy lifting and the securing of mathematical bounds, but then definitively by Vershik and Kerov [9], that asymptotically  $\langle \ell_n \rangle \rightarrow 2\sqrt{n}$ . Indeed, this was well presented by Baer and Brock; see Fig. 1 and Table 2 of their early paper, where they recorded exact enumerations of the LIS PDF for  $n \leq 36$  using the RSK correspondence, hook formula, and Young tableau mapping [10], complementing these results with approximate Monte Carlo data reaching up to  $n = 10^4$ . Later, it was conjectured by Odlyzko and Rains, then firmly established by Kim, that the standard deviation about this mean scaled as  $n^{1/6}$ ; i.e., the Ulam-LIS problem possessed a well-defined limit distribution in the variable  $(2\sqrt{n} - \ell_n)/n^{1/6}$ . Baik, Deift, and Johansson then proved rigorously that the underlying, asymmetric non-Gaussian distribution was, surprisingly, TW-GUE from random matrix theory [11]. In fact, Odlyzko and Rains provided explicit Monte Carlo evidence in this regard, matching their semilog LIS PDF, with  $10^5$  permutations of length  $n = 10^6$ , against a precise Painlevé II rendering of Tracy-Widom GUE. Interestingly, the Odlyzko-Rains data show a small, but clear, finite-time offset (they quote  $\langle \chi_2 \rangle \approx -1.720$ ) in their LIS PDF, which only stubbornly disappears as the authors heroically head toward asymptopia, pushing their simulations to  $n = 10^8$ , where  $\langle \chi_2 \rangle \approx -1.758$ , and further ( $n = 10^{10}$ ), nearly closing the gap between themselves and the known TW-GUE universal mean:  $\langle \chi_2 \rangle = -1.7711$ . Since  $\beta_{\text{LIS}} = \frac{1}{6}$ , random permutations of  $n = 10^{10}$  digits are equivalent, roughly, to generating pt-pt DPRM with  $t = 10^5$ ; quite a demanding task, indeed. Here, we take a different attitude, considering rather short permutation strings, with  $n = 10^2$ – $10^4$ , essentially the regime where Baer and Brock concentrated their numerical enterprise, but invoke KPZ scaling wisdom to, nevertheless, pin down very precise values for the moments. See Fig. 1, where our Ulam-LIS PDF follows from  $10^8$  random permutations of length  $n = 10^4$ , as well as the figure insets, which document the KPZ scaling of the truncated LIS cumulants, allowing us to make fine asymptotic estimates:  $(-1.7715, 0.8135, 0.2245, 0.091\,02)$ , close to known TW-GUE values. For the Ulam-LIS variance, skewness, and kurtosis (lower inset) we have invoked an  $n^{-2\beta}$  finite-“time” correction, suggested by DPRM lore. The upper inset, which examines KPZ scaling of the universal TW-GUE mean, amends the Odlyzko-Rains ansatz to include an additive constant term, resulting in an  $n^{-\beta}$  finite-time correction to  $\langle \chi_2 \rangle$ , plainly visible in the data. In our Ulam-LIS work, we have come to refer to this quantity as Baer-Brock’s constant, here estimated to be  $\approx 0.51 \pm 0.01$ , given by the slope of the line in our plot of  $\langle \chi_2 \rangle$  versus  $n^{-1/6}$ . The existence of such a *model-dependent additive constant* is well known to KPZ practitioners [21,29,31,32], but has not, to our knowledge, been discussed for Ulam-LIS. Given the somewhat privileged role of the random permutation problem amidst the menagerie of systems obeying TW-GUE statistics—it is, by the hook formula, uniquely accessible to exact enumeration for finite

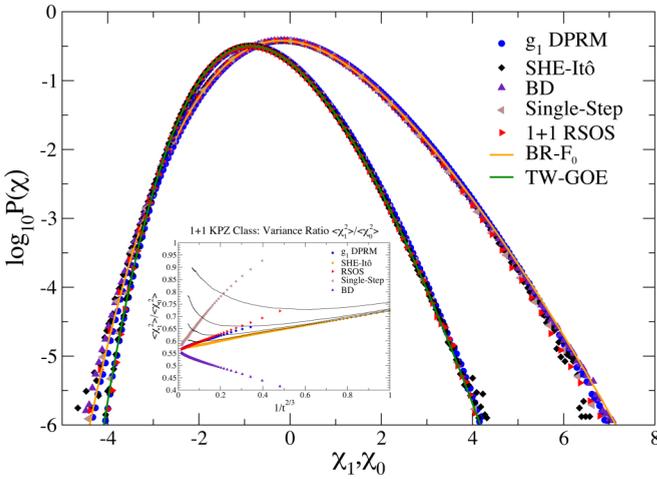


FIG. 2. (Color online)  $1 + 1$  transient-regime and stationary-state KPZ/DPRM class: Comparison to Tracy-Widom GOE and Baik-Rains  $F_0$  limit distributions. Inset: Universal variance ratio as precursor signature of KPZ SS statistics. Black traces document crossover scaling to the stationary state, accessible via experiment.

$n$ —we attach somewhat greater significance to this particular additive constant.

In Fig. 2, for our  $g_1$  DPRM, single-step growth, and SHE-Itô Eulerian integration, we have paired up the distinct PDFs associated with the *transient-regime* and *stationary-state* statistics for the flat KPZ geometry, making relevant comparisons to exact Tracy-Widom GOE and Baik-Rains  $F_0$  distributions, respectively. For the *transient* dynamics, our DPRM/SHE work harks back to early, pre-TW efforts [33,34], and dovetails with recent studies [35], in which several KPZ kinetic roughening PDFs—among them BD and RSOS—are *fit* to TW-GOE. As an indication of our findings here, we note specifically that the  $g_1$  DPRM yields a variance, 0.637 03, quite close to the TW-GOE result  $\langle \chi_1^2 \rangle = 0.638 05$ , along with a skewness, 0.2976, very much in line with the value,  $s_1 = 0.2935$ , characteristic of that distribution. More importantly, with regard to  $(1 + 1)$  KPZ-class *stationary-state* statistics, and comparison to the Baik-Rains limit distribution, our DPRM/SHE results, well matched by single-step, BD, and RSOS findings, convincingly establish universality of the seminal PNG studies of Prähofer and Spohn [19]—note the strong data collapse of our models upon the BR- $F_0$  trace, a highly nontrivial numerical matter. For the  $g_1$  DPRM, we have evolved the system to a late time  $t_o = 8 \times 10^4$ , then study dynamic temporal correlations in the path free-energy  $\Delta F(t) = F(x, t_o + t) - F(x, t_o)$ , during the subsequent time interval  $t = 500$ , averaging over  $8 \times 10^4$  runs in a system of size  $L = 10^4$ . At  $t = 500$ , we note the approximate values (1.1177, 0.3432, 0.252) which, extrapolated to infinite  $t$ , yield (1.1466, 0.3530, 0.278), quite close to the known [19] BR- $F_0$  variance, skewness, and kurtosis:  $(\langle \chi_0^2 \rangle, s_0, k_0) = (1.150 39, 0.359 41, 0.289 16)$ . For RSOS growth in the stationary state,  $t_o = 40k$  and  $t = 240$ , which gives an asymptotic estimate  $\langle \chi_0^2 \rangle \approx 1.1331$  for the Baik-Rains constant, while ballistic deposition, with  $t_o = 50k$  and  $t = 500$ , produces 1.1382 for this quantity. We emphasize that the plotted distributions of Fig. 2 are not the raw data, but have been rescaled, each in turn, to represent the *system-dependent*

asymptotic variances, carefully extracted via KPZ scaling theory [12,22,24].

With an inset to Fig. 2, we consider the time dependence of the *ratio of variances* in KPZ transient and stationary-state regimes. We are motivated here by Takeuchi's recent study [16] of the PNG model, as well as his attempts to distill from the  $1 + 1$  KPZ-class turbulent liquid-crystal data [17], some sign of the SS statistics. Although the BR- $F_0$  limit distribution itself remained well beyond the reach of these experiments, Takeuchi discovered an impressive precursor signature of the KPZ stationary-state, evident as a *skewness minimum*, which we confirm below with our own models, establishing its universal KPZ aspect. Here, we propose an alternative signature, relying upon a more statistically tame quantity which, asymptotically, is fixed universally:

$$\lim_{t \rightarrow \infty} (\delta F^2(t))_{\text{tran}} / (\Delta F^2(t))_{\text{SS}} = \frac{\langle \chi_1^2 \rangle}{\langle \chi_0^2 \rangle} = 0.554 64$$

by the ratio of the TW-GOE variance and Baik-Rains constant. Since the *variance ratio*, like the skewness  $s$ , requires no knowledge of  $\Gamma$ , a quick examination of this quantity in experiments may provide a glimpse of a key universal property of the KPZ stationary-state. Within the Fig. 2 inset, we document (solid symbols) the scaling of this interesting ratio for  $g_1$  DPRM, SHE-Itô, and three distinct KPZ growth models. All five systems converge convincingly—even the contrarian BD, well known for its curmudgeonly behavior [36]. For the SHE-Itô case,  $t_o = 10k$  and  $t = 100$  as above, per our rendering of the Baik-Rains distribution. However, we also include within the inset, data sets associated with lesser values of  $t_o$  and hundredfold fewer runs, to study *crossover effects*, indicating how this scaling phenomena might appear in a more constrained experimental context. The black traces, top down, correspond to  $t_o = 3, 20, 100, 500$ , though all possess  $t = 100$ , with statistical averaging now done [37] over  $10^6$ , rather than  $10^8$  points. The goal here is to provide an extra tack upon the  $1 + 1$  KPZ-class stationary-state statistics, and to further complement the spectacular experimental results already in hand for the transient *flat* TW-GOE and *curved* TW-GUE KPZ classes.

Lastly, with Fig. 3, we establish *universality* of Takeuchi's skewness minimum [16], heightening its utility as an experimental precursor signature of the  $1 + 1$  KPZ-class stationary-state statistics. For our Gaussian DPRM, SHE-Itô integration, and KPZ growth models, we have plotted up the fluctuation PDF skewness  $s(\tau)$  as a function of the dimensionless scaling parameter  $\tau = \Delta t / t_o$  where, in each case, the numerics are evolved to a time  $t_o$ , the skewness then tracked through a subsequent time interval  $\Delta t$ . To consider the consequences of an *insufficiently mature* kinetically roughened state, we include SHE-Itô results for  $t_o = 25, 50$ , and  $100$ , respectively, shown in ascending, dashed curves, and solid in the last instance. It is interesting to observe how, with increasing  $t_o$ , the traces develop to reveal a nearly full-fledged Takeuchi minimum; i.e., for  $t_o = 25$ , the skewness minimum is entirely absent, though the curve possesses a suggestive inflection point. At  $t_o = 50$ , the minimum has appeared, but only gains a fuller expression at  $t_o = 100$ . The evolution, with increasing  $t_o$ , of the universal KPZ-class skewness minimum is tied primarily

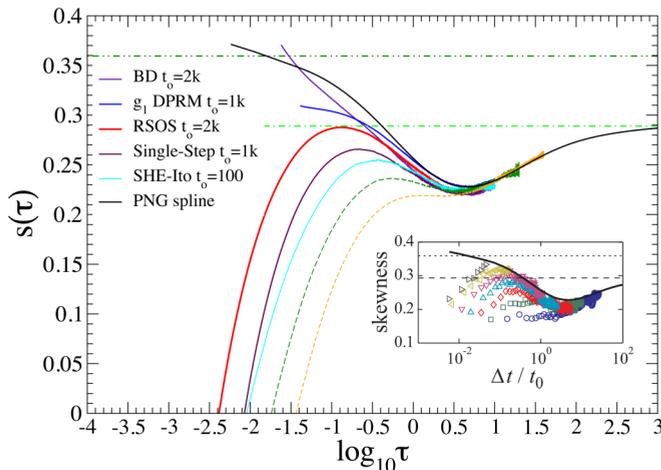


FIG. 3. (Color online) Takeuchi *skewness minimum*, evident in our DPRM, SHE-Itô, BD, RSOS, and single-step kinetic roughening models. Upper and lower dot-dashed lines correspond to asymptotic skewness,  $s_0 = 0.35941$  and  $s_1 = 0.2935$ , of the Baik-Rains  $F_0$  and Tracy-Widom GOE limit distributions, respectively. Inset: Data sets, KPZ turbulent liquid-crystal experiments (Ref. [16]). The emergence of the minimum for our SHE-Itô integration, revealed as  $t_o$  grows from 25, 50, to 100, is manifest within the experiment itself, where the symbols indicate distinct times, smallest to largest,  $t_o = 2, 6, 10, 18, 30, 54, 60$ , right to left.

to its *curvature* and, in the  $(1+1)$ -dimensional case, to the steepening slope for  $\tau \lesssim \tau_{TM}$ , as  $t_o \rightarrow \infty$ . We mention, in passing, that our SHE-Itô, single-step, and RSOS simulations share a common *nonuniversal behavior* at very early  $\Delta t$ , which gives rise to a *model-dependent* maximum and subsequent ( $\tau \ll 1$ ) monotonic decrease in  $s(\tau)$ , evident in the figure. Most interestingly, this peculiar feature is actually manifest within Takeuchi's [16] liquid-crystal KPZ experimental data set proper, shown as an inset within Fig. 3 and contrasts, intriguingly, with his own PNG numerical results, which indicate a weakly divergent, *increasing*  $s$  for the smallest  $\tau$

values [38]. These system-dependent details are small, but curious matters, slightly off-stage, but revealing nonetheless. From an experimental point of view, however, the Takeuchi minimum itself is the essential focus and sits center stage, providing a *crucial, telltale precursor signature of 1+1 KPZ-class stationary-state statistics*. Hence, in Fig. 3, our DPRM/SHE results, set alongside KPZ kinetic roughening models. Included, too, in both the figure and inset, is Takeuchi's skewness spline (black traces), carefully crafted via his PNG work. The agreement between Takeuchi's spline and our mix of KPZ data sets, and  $g_1$  DPRM in particular, is quite suggestive. From our simulations, we locate the Takeuchi minimum at  $s_{TM} = 0.225 \pm 0.005$ , for  $\tau = \frac{\Delta t}{t_o} = 4.5 \pm 0.4$ . Note, it is only in the double limit,  $t_o \rightarrow \infty$ , then  $\Delta t \rightarrow \infty$ , with  $\tau = \Delta t/t_o \rightarrow 0$  that an interpolating skewness spline, originating at the TW-GOE value,  $s_1 = 0.2935$ , for large  $\tau \gg 1$ , then exhibit the correct asymptotic approach to the stationary-state Baik-Rains value,  $s_0 = 0.35941$ , for vanishing small  $\tau \ll 1$ . To manifest the latter would be quite difficult, even numerically. Finally, we mention that the *kurtosis minimum* we find for the  $g_1$  DPRM, at  $k_{TM} = 0.117$ , is quite near Takeuchi's PNG value for this more demanding, higher cumulant ratio. Eliciting such a delicate feature experimentally will remain, no doubt, a most challenging task. In the interim, we await a clever experimental implementation of the KPZ stationary-state initial condition, which might allow *direct* access to Baik-Rains  $F_0$ . Given the universality established via Fig. 2, with our DPRM/SHE, single-step, BD, and RSOS results, this limit distribution represents, certainly, the relevant fixed point PDF.

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