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Directed polymers versus directed percolation

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Universality plays a central role within the rubric of modern statistical mechanics, wherein an insightful continuum formulation rises above irrelevant microscopic details, capturing essential scaling behaviors. Nevertheless, occasions do arise where the lattice or another discrete aspect can constitute a formidable legacy. Directed polymers in random media, along with its close sibling, directed percolation, provide an intriguing case in point. Indeed, the deep blood relation between these two models may have sabotaged past efforts to fully characterize the Kardar-Parisi-Zhang universality class, to which the directed polymer belongs. [S1063-651X(98)50810-0]

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Condensed matter physicists have become increasingly aware of the complex fractal geometric properties and rich nonequilibrium statistical mechanics exhibited by stochastic growth models, such as diffusion-limited aggregation (DLA) and Eden clusters. In fact, the history of DLA [1], now some twenty years old, provides an interesting cautionary tale for those in the community concerned with far-from-equilibrium dynamics. Although well-characterized numerically, and a compelling discrete model for various physical processes, such as electrochemical deposition [2], viscous fingering in porous media [3], and starved bacterial growth [4], DLA remains stubbornly shackled to the lattice formulation at its essence, with various theoretical formulations based upon a continuum description making slow, difficult progress. A rich contrast is provided by various *self-affine fractals*, such as Eden clusters, ballistic deposits, polynuclear growth, etc., that fall within a single, rather large universality class governed by the celebrated stochastic partial differential equation of Kardar, Parisi, and Zhang (KPZ) [5]. In this case, there is quite extraordinary success of the continuum theory, shedding much light upon numerical experiment. Connection to actual physical systems has, however, been a bit less forthcoming. Even so, patience has apparently paid off, since there have been recent papers on diverse applications, such as flame-front propagation [6], cumulus cloud formation [7], and matter correlation in the universe [8], all suggesting strong evidence of KPZ behavior. Interest in the KPZ equation is amplified many fold by its mathematical relation to directed polymers in random media (DPRM) [9], a baby version of the spin-glass problem and one of the few exactly solvable problems of ill-condensed matter. In addition, there is an explicit link to the stochastic Burgers equation as manifested in the simple asymmetric exclusion process, beautifully captured in the work of Derrida *et al.* [10]. With kinetic roughening phenomena, DPRM, and the stochastic Burgers equation constituting the so-called KPZ triumvirate, serious capital investment has been made toward understanding the fundamental properties of this nonlinear stochastic partial differential equation (PDE), as well as the models that fall within its purview.

In this Rapid Communication, we discuss the difficult legacy that the DPRM inherits via directed percolation (DP). Simulations are done in the wedge geometry, basic numerical details found in a recent review paper [11]. In a nutshell, however, we consider the first quadrant of the square lattice, rotated 45°, with random energies placed upon the diagonally oriented bonds. Directed paths up to the nth time slice have $n+1$ possible endpoints, the total energy of a path given by the sum of the bonds visited along the way. Due to the 2^{n+1} configurations accessible to the DPRM, an entropically generated elasticity competes against energy gains associated with particularly favorable bonds in the bulk, leading to a *superdiffusive* wandering exponent ζ, characterizing transverse wandering, $|x|\sim t^{\zeta}$, of the DPRM. In $d=1$, $\zeta=2/3$ exactly, by virtue of a fluctuation-dissipation theorem. Functional renormalization group methods [12] have led to the conjecture $\zeta(d)=6/(d+8)$, which retrieves this known result, captures a wandering exponent decreasing with dimensionality and, interestingly, suggests the possibility of a finite upper critical dimension (UCD), $d_c=4$, which appears to be corroborated by various KPZ mode coupling [13], DPRM field-theoretic [14], and directed percolation [15] arguments. It should be stressed that KPZ/DPRM numerical work, including real-space renormalization group techniques [16], transfer matrix [17], and growth-model simulations [18], shows no evidence of a finite UCD, a synopsis of best efforts to date indicating $\zeta=0.624$, 0.60, 0.57, 0.54 for $d=2,3,4,5$. Even so, it is readily apparent that four is a very special dimension, indeed, for KPZ strong-coupling physics, as indicated by the glassy dynamics unearthed by Moore *et al.* [19], and the complete breakdown of perturbation theory recorded by Wiese's exact calculation [20] of the DPRM β function. In fact, as we will argue below, KPZ glassiness is associated with the close kinship of the DPRM problem to directed percolation, and manifests itself in even lower dimensions. Finally, this blood tie lies at the heart of purported KPZ *nonuniversality* advertised recently by Newman and Swift [21] and, furthermore, reveals that there is still much to learn regarding the DPRM problem, even in $d=1$.

Although it is well known that both spatially correlated [22] and uncorrelated, but long-tailed power-law [23] noise can drastically alter the DPRM wandering exponent, we stress that the distributions under consideration here are finite, bounded, well-defined, and seemingly innocuous. There are no built-in spatial correlations and all moments are finite. However, the distributions are either tied directly or bear a strong relation to the DPRM's blood sibling, directed percolation, and, given the chance, they will readily reveal their lattice roots. As such, these distributions can easily violate any hypothesis of KPZ universality, since such notions are based, traditionally, upon an all-consuming continuum description; in this case, a single governing stochastic PDE that transcends microscopic details, such as the lattice. Nevertheless, as we have learned the hard way for DLA, shedding the lattice may be a nontrivial matter. Here, we discover that the discreteness of directed percolation is nearly as difficult for the DPRM to disown.

Consider the following bimodal (BMD) DPRM, which has random bond energies $\varepsilon=-1,+1$ drawn with probabilities $p,(1-p)$, respectively [24]. For the time being, we restrict ourselves to $d=1$ and examine the system at zero temperature, where DPRM statistical mechanics becomes a matter of global optimization; that is, we search for the directed path of least energy, traversing the length of the random energy landscape. In Fig. 1(a), we show a double-log plot of the rms energy fluctuations of the BMD DPRM for various values of $p<p_c\approx 0.645$, so that we are below the -1 percolation threshold for this dimensionality; i.e., the energetically, more desirable negative energy bonds do not percolate. For each value of p, we ensemble average over 200 000 realizations of disorder, for paths of 1000 steps. Not surprisingly, for $p=0.25$, which is well below threshold, we quickly retrieve the anticipated exact value $\omega=(\zeta-1)/2=1/3$, which is the early-time roughness exponent β in the KPZ growth model context. A least-squares fit to the final 500 points yields 0.324. By contrast, for $p=0.65$, just above threshold for -1 bonds, the energy fluctuations actually saturate ($\Rightarrow\omega=0$) thanks to the presence of percolating clusters of -1 bonds, which bring with them a great ground-state degeneracy. Interestingly, as p increases, the fitted value of ω decreases, indicating an increasingly greater role played by finite spanning clusters that will exist with ever greater probability as p approaches the thermodynamic threshold from below. Figure 1(b) provides quantitative insight into the BMD DPRM scaling behavior, showing the *effective* energy exponent ω_{eff} as a function of $t^{-1/3}$. For $p=0.25$, the curve is essentially flat, heading stage left for the value 1/3. For $p=0.52$ and 0.56, the curves start from a lower point and then, following a temporary initial drop, steadily increase as expected. Even so, a fit to the latter, as above, produces an estimate for $\omega\approx 0.295$ that is still more than 10% too low.

Of course, there is little worry that the exact value 1/3 will be recovered in the thermodynamic limit. Things become increasingly more severe, though, for $p=0.60$, 0.62, and finally 0.64, just 0.8% below threshold, where the effective exponent decreases towards zero before finally turning round to begin a slow climb in its final 500 steps. Nevertheless, at the end, it has still only managed to reach $\approx 0.12^+$, quite distant from its presumed asymptotic value. The behavior for $p=0.62$, 3.9% below threshold suggests, however, that the climb, though slow, will stubbornly continue, a victim of a crossover length scale strongly dependent upon the difference $(p-p_c)$. In a lesson learned long ago, in the context of equilibrium critical phenomena, the RG's approach to asymptopia can be strongly dependent upon the initial distance

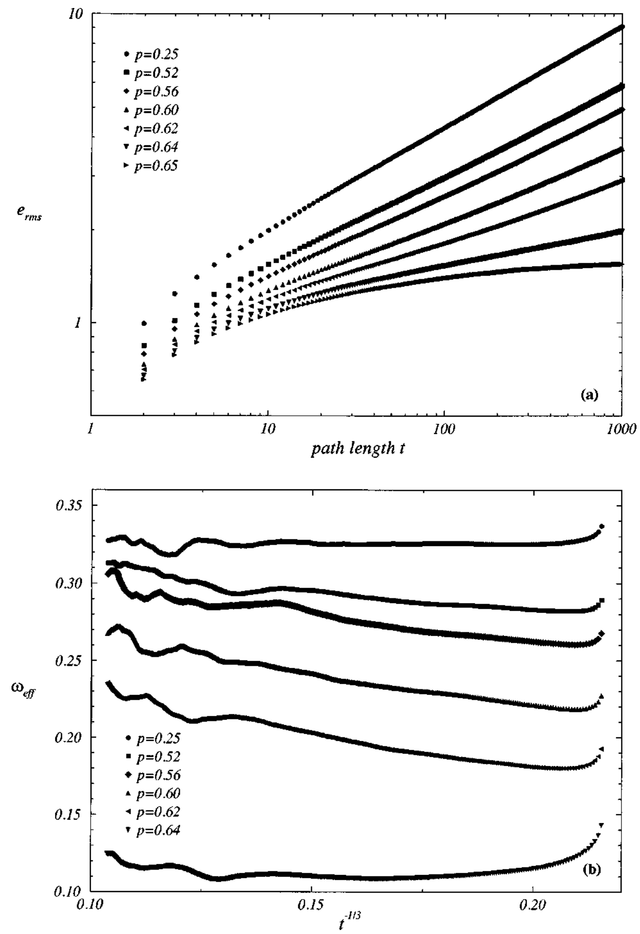

FIG. 1. (a) RMS energy fluctuations of the $1+1$ BMD DPRM, and (b) associated *effective* energy exponent.

from the fixed point; i.e., chosen values for the bare parameters. For the BMD DPRM at hand, this wisdom translates very simply. If you make a bad choice for your bare bond-energy probability distribution (e.g., $p>p_c$), KPZ universality is moot, because you flow to an altogether different fixed-point function (FPF), that of directed percolation. For a choice sufficiently far from that boundary ($p\ll p_c$), the DP legacy is negligible and we flow undeterred to asymptopia. For $p\to p_c$, however, a lengthy detour ensues, much influenced by the neighboring subspace defining the DP basin of attraction.

Interestingly, these findings have strong ramifications for a recent suggestion [21] regarding possible KPZ nonuniversality. It may be thought that the above behavior is an artifact of the particular discrete bimodal distributions that we've chosen, characterized as they are by the single parameter p. In fact, the results are quite general and have important implications for all simulations done to extract DPRM critical indices [25]. Moreover, because p_c decreases with dimensionality (e.g., $p_c\approx 0.382$ for the $2+1$ directed bond percolation on a simple cubic lattice [26]), the DP FPF gains greater influence as d is increased, undermining numerical efforts to determine unadulterated, asymptotic exponents. The situation can be exacerbated by a particularly poor choice of bare probability distribution. For example, a flat distribution, the lazy man's favorite, is fine for the $1+1$ DPRM, but does increasingly worse in higher dimensions.

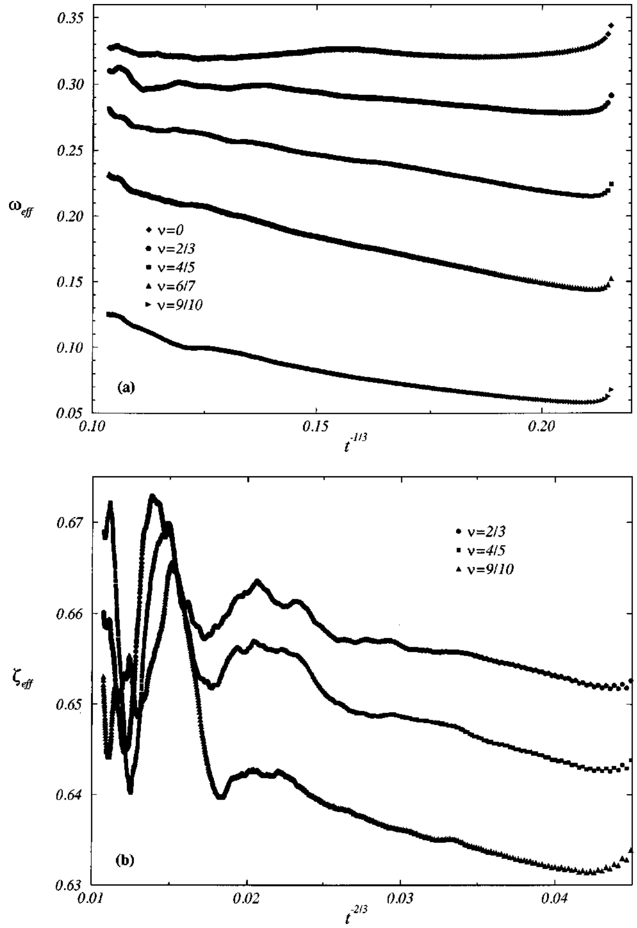

FIG. 2. Effective (a) energy, and (b) wandering exponent for $1+1$ SBC DPRM.

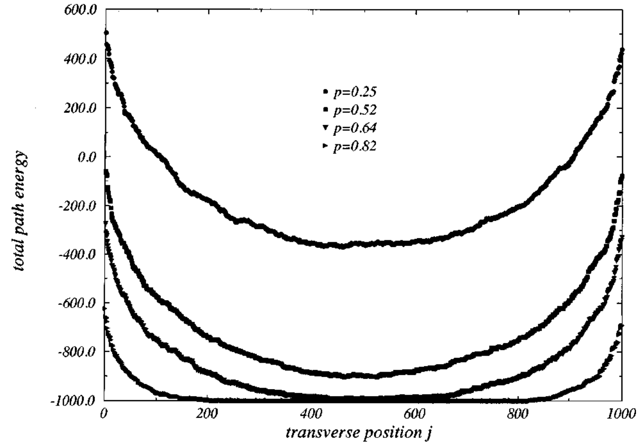

FIG. 3. Terminal energy profile of $1+1$ BMD DPRM; single realization of the random energy landscape. Note the condensation of ground-state degeneracy as $p \rightarrow p_c$.

The effect is well documented [17,21] and easy to see, yet it remained entirely inexplicable, at least until now. There is simply *too much integrated weight in the low end of the distribution*. As dimensionality is increased, this bare distribution is pushed further from the inevitable DPRM FPF and closer to, but never within, the DP basin of attraction, thereby lengthening the resulting trip to asymptopia. Since CPU time is fixed, larger d means smaller system sizes and, to the simulator's chagrin, it becomes a lose-lose situation for the uniform distribution. The Gaussian is always a safe bet, with negligible weight at the low end, but it is not necessarily the most clever choice. In related work [27], we establish criteria for an optimally selected bare distribution that readily yields the true many-dimensional DPRM wandering exponent.

An especially disastrous choice, even in $d=1$, is the strongly biased continuous (SBC) distribution, given by $P(0<\varepsilon<1)=\varepsilon^{-\nu}$, which possesses finite moments of arbitrary order, provided that $\nu<1$. In Fig. 2(a), we illustrate our findings for ω_{eff} in zero-temperature simulations of this SBC DPRM. For $\nu=2/3$ the results are little different from the flat distribution ($\nu=0$), whose curve is nearly level, reminiscent of the large p limit of the BMD DPRM, see Fig. 1. The effective energy exponent varies little from 1/3, with no noticeable effect of the DP FPF. As ν is increased to unity, however, and more probabilistic weight is thrown into the low end, ω_{eff} suffers a dramatic decrease. In fact, the apparent nonuniversality exhibited by the $\nu=9/10$ SBC DPRM is strikingly similar to the that of the $p=0.64$ BMD DPRM, suggesting that simulations done on (presumably prohibitively) larger system sizes would permit the former to recover the asymptotic value 1/3. Finally, an examination of ζ_{eff} for the SBC DPRM shows that, for extreme biasing, the wandering exponent initially decreases to ≈ 0.63 in the early stages of the walk [see Fig. 2(b)], a vestigial link to DP's $\nu_\perp/\nu_\parallel=0.6326\pm0.0002$ in this dimensionality. This finding may have repercussions for the statistics of fractal tear lines in paper [28], a physical realization of the $1+1$ DPRM, but one in which dynamic aspects, associated with percolative paths, may have important effects.

A characteristic feature of the BMD DPRM is the onset of an ever-increasing ground-state degeneracy as p increases above p_c. The globally optimal path of least energy is no longer unique; rather, one obtains for a single realization of the random energy landscape, a *minimal cluster*, which corresponds to a directed lattice animal with multiple paths of equal energy. Of course, below p_c, there is always a single such trajectory and the free-energy profile, when averaged over many realizations of disorder, exhibits its tell-tale quadratic form [11,29], centered about the minimal value of the diagonal direction. In Fig. 3, we show the evolution of the $1+1$ BMD DPRM free-energy profile, as a function of transverse position along the final time slice, for increasing values of p. As is apparent from the figure, the profile becomes progressively flatter as $p \rightarrow p_c$, at which point the curve acquires a discontinuous first derivative, and consists of a flat segment dead center, plus quadratic wings. An examination of the free-energy profile for the $1+1$ SBC DPRM reveals manifestation of a similar phenomenon as $\nu \rightarrow 1$, although the effect is only suggestive, since a continuous distribution precludes the possibility of true ground-state degeneracy. Even so, the perturbing influence of BMD DPRM degeneracy may manifest itself as a transient glassy regime in the original model.

As an alternative indication of the strong effect of the directed percolation FPF, we show, in Fig. 4, the disorder-averaged free-energy probability distribution function (PDF) for the $1+1$ BMD DPRM, corresponding to various values of p below the percolation threshold. Note that these refer to

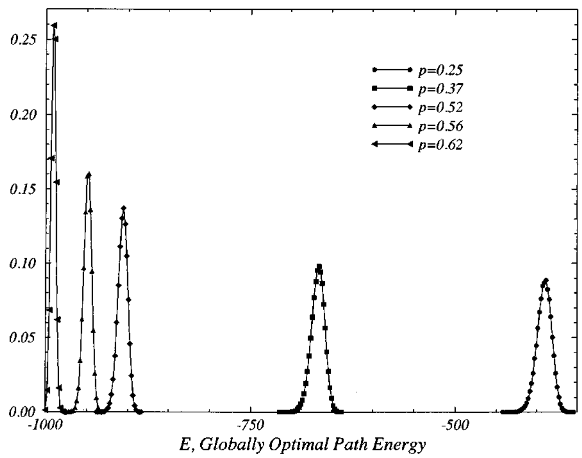

FIG. 4. Disorder-averaged free-energy probability distribution function for $1+1$ BMD DPRM.

the total energy of the thousand-step walks, so $E=-1000$ is the extremal value of the abscissa. For $p>p_c$, the free-energy PDF is a δ-function spike at this value. However, for $p<p_c$, the -1 bonds do not percolate and the PDF is pushed off this point. A standard measure of KPZ/DPRM universality is the *skewness* of this distribution, which is known [17,30] to be negative, signature of the global optimization at work, and to have the value $\approx -0.29\pm 0.02$ in this dimensionality. For $p=0.25$, the rightmost curve, we find -0.274 for this parameter, in keeping with the standard DPRM result. However, as p rises toward threshold, $p=0.37$, 0.52, 0.56, 0.62, we find diminishing skewness, -0.263, -0.189, -0.161, -0.074, respectively, as the free-energy PDF slides left and is nearly transformed into the δ-function spike characteristic of the directed percolation FPF. In fact, for $p=0.64$, which is still below threshold though yielding a startlingly small effective energy exponent [Fig. 1(b)], the skewness has become sizeably positive, $+0.484$. This state of affairs is inevitable, since $E<-1000$ is ruled out and the ensemble averaging is dominated by globally optimal paths that are almost entirely -1 bonds, with one or two $+1$ bonds located somewhere early on in the trajectory. The behavior of the bond-energy PDF is actually quite similar for the SBC DPRM; e.g., for $\nu=4/5$, we find a skewness, -0.051, that is nearly zero. More extreme biasing, $\nu=9/10$, produces positive skewness, leaving little doubt about the strong DP-DPRM connection.

In summary, we have explored the close blood relationship between DP and DPRM problems, discovering why, in the case of the latter, certain bare bond-energy PDFs are superior, while others are outright damning. That these effects are already severe in $d=1$ is somewhat surprising. Nevertheless, as dimensionality increases, the expanding DP basin of attraction exerts ever greater influence upon transient DPRM scaling, thanks to smaller percolation thresholds. An intriguing thought concerns reversing this power struggle; i.e., how might the DPRM affect a more broadly defined DP problem? Might this transcendent model then provide the link for the DP and DPRM to share a common UCD, $d_c=4$?

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